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RADIATIVE HEAT TRANSFER BETWEEN MOVING SURFACES

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Abstract—The equations of radiative heat transfer between surfaces in relative motion are derived in this paper. A particular case of parallel plates is studied as an example. The solution is obtained by numerical methods.

NOMENCLATURE

A_1, A_2, absorptivity of surface 1 or surface 2, respectively;
b, thickness;
c, specific heat;
E_1, E_2, emitted radiation flux of surface 1 or surface 2, respectively;
F_1, F_2, heat-transfer area;
F_{1-2}, mutual heat-transfer area;
H_1, H_2, irradiations;
J_1, J_2, brightnesses;
k, thermal conductivity;
l, distance between the plates (see Fig. 1);
m, parameter, equation (20);
n, parameter, equation (22);
$\dot{Q}$, heat rate;
q, local heat flux;
q_m, mean heat flux;
R_1, R_2, reflectivities;
T_1, T_2, absolute temperatures;
U, energy;
w, velocity;
x, y, z, co-ordinates;
x^+, dimensionless co-ordinate, equation (20).

Greek symbols

γ, specific gravity;
ϵ_1, ϵ_2, emissivities;
θ, dimensionless temperature difference, equation (23);
ϑ, dimensionless temperature, equation (20);
ξ, co-ordinate;
σ, Stefan–Boltzmann constant;
τ, time;
ϕ, elementary configuration factor, equation (3) and Fig. 1;
ϕ_{1-2}, local configuration factor;
Ω, perimeter.

THE cases of radiative heat transfer, in which the heat exchanging surfaces are in relative motion, are of conspicuous practical importance. However, such problems (as far as the author is informed) have not been analysed since. An example of such a problem and the method of its solution is to be shown in this paper.

The considered system consists of two surfaces assumed to be gray and diffusely reflecting, one of which is at rest and the other moves with constant velocity w.

Let J_1, J_2 be the local brightnesses of the surfaces; then $d^2\dot{U}_{1-2} = J_1\, d^2F_{1-2}$ is the decrease of energy of the element dF_1 of surface F_1, caused by radiation towards the element dF_2 of surface F_2. d^2F_{1-2} denotes the mutual area of the elements dF_1, dF_2.

On the other hand, the element dF_2 radiates towards dF_1 a portion $d^2\dot{U}_{2-1} = J_2 \,.\, d^2F_{1-2}$. Therefore, in the state of equilibrium, a heat rate $\dot{Q}$ is supplied to the surface F_2, and

$$d^2\dot{Q} = d^2\dot{U}_{1-2} - d^2\dot{U}_{2-1} = (J_1 - J_2)\, d^2F_{1-2},$$

or, after integration in respect of surface F_1

$$d\dot{Q} = \int_{F_1} (J_1 - J_2)\, d^2F_{1-2}. \qquad (1)$$

The local heat flux at the surface F_2 is thus

$$q = \frac{d\dot{Q}}{dF_2} = \int_{F_1} (J_1 - J_2) \,.\, \frac{d^2F_{1-2}}{dF_2},$$

or

$$q = \int_{F_1} (J_1 - J_2)\phi \, \mathrm{d}F_1, \tag{2}$$

where

$$\phi = \frac{\cos \beta_1 \, . \, \cos \beta_2}{\pi r^2} = \frac{\mathrm{d}^2 F_{1-2}}{\mathrm{d}F_1 \, . \, \mathrm{d}F_2}. \tag{3}$$

The brightnesses J_1 and J_2 may be evaluated from

$$\left.\begin{aligned} J_1 &= E_1 + R_1 H_1, \\ J_2 &= E_2 + R_2 H_2, \end{aligned}\right\} \tag{4}$$

where E_1, E_2 are the emitted radiation fluxes, R_1, R_2—the reflectivities, and H_1, H_2—the irradiations.

The latter are defined by

$$\mathrm{d}H_1 = \frac{\mathrm{d}^2 U_{2-1}}{\mathrm{d}F_1}, \qquad \mathrm{d}H_2 = \frac{\mathrm{d}^2 U_{1-2}}{\mathrm{d}F_2},$$

wherefore, in connection with (3), we obtain

$$H_1 = \int_{F_2} J_2 \phi \, \mathrm{d}F_2, \qquad H_2 = \int_{F_1} J_1 \phi \, \mathrm{d}F_1. \tag{5}$$

Elimination of the irradiations from (4) and (5) yields

$$J_1 = E_1 + R_1 \int_{F_2} J_2 \phi \, \mathrm{d}F_2,$$

$$J_2 = E_2 + R_2 \int_{F_1} J_1 \phi \, \mathrm{d}F_1,$$

or

$$\left.\begin{aligned} J_1 = E_1 &+ R_1 \int_{F_2} E_2 \phi \, \mathrm{d}F_2 \\ &+ R_1 R_2 \int_{F_2} \mathrm{d}F_2 \int_{F_1} \mathrm{d}F_1 \, . \, \phi^2 J_1, \\ J_2 = E_2 &+ R_2 \int_{F_1} E_1 \phi \, \mathrm{d}F_1 \\ &+ R_1 R_2 \int_{F_2} \mathrm{d}F_2 \int_{F_1} \mathrm{d}F_1 \, . \, \phi^2 J_2. \end{aligned}\right\} \tag{6}$$

The difference of the brightnesses $\Delta J = J_1 - J_2$ satisfies the integral equation

$$\Delta J = E_1 - E_2 + R_1 \int_{F_2} E_2 \phi \, \mathrm{d}F_2 - R_2 \int_{F_1} E_1 \phi \, \mathrm{d}F_1 + R_1 R_2 \int_{F_2} \mathrm{d}F_2 \int_{F_1} \mathrm{d}F_1 \phi^2 \Delta J. \tag{7}$$

Now, let the surface F_1 be a radiator of constant temperature T_1 and emission $E_1 = \epsilon_1 \sigma T_1^4$, where ϵ_1 is the emissivity of that surface. The surface F_2 is heated by radiation. Let us assume that the surfaces are cylindrical with arbitrary profile, and their generatrices are parallel to each other and to the direction of the velocity w. With the notations from Fig. 1 we may write down the equation of heat conduction in the body 2 with surface F_2, viz.

$$c\gamma\left(w \frac{\partial T}{\partial x} + \frac{\partial T}{\partial \tau}\right) = k\left(\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + \nabla_1^2 T\right), \tag{8}$$

where ∇_1^2 is the Laplacian operator in the plane perpendicular to the axis x, which is parallel to the direction of motion (and to the generatrices); the symbols c, γ and k denote the specific heat, the specific gravity and the heat conductivity of the body 2.

If $w \neq 0$ the problem may be stationary, $\partial T/\partial \tau = 0$, and the boundary conditions for equation (8) are:

$$\begin{aligned} T &= T_2, \\ k \, . \, \mathrm{grad}\, T &= q \end{aligned} \tag{9}$$

at the surface.

The system (2), (7), (8) and (9) is sufficient to evaluate the temperature T, and consequently T_2, which is variable in the x-direction, whereas T_1

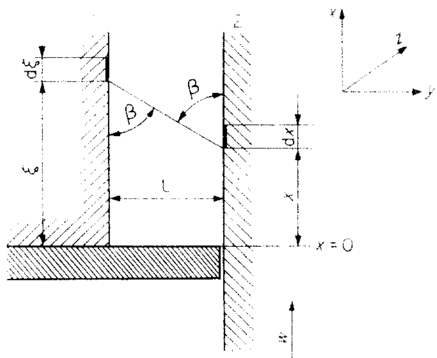

FIG. 1.

is assumed to be constant. As a supplementary initial condition we may assume that at the entrance (see Fig. 1) $x = 0$, it is $T = T_2 = T_{2,0} < T_1$. Consequently, for $x \to \infty$ it must be $T = T_2 = T_1$.

The problem thus formulated is very complicated. We will solve a particular simplified case by assumption that the body 2 is a sheet of (small) thickness b. The temperature across the section in the yz-plane may be thus regarded as constant.

From integration of (8) over the area F of the cross-section in the yz-plane we get

$$c\gamma w F \frac{\mathrm{d}T_2}{\mathrm{d}x} = k\left(F \, . \, \frac{\mathrm{d}^2 T_2}{\mathrm{d}x^2} + \int_F \nabla_1^2 T \, \mathrm{d}F\right)$$

for a stationary case.

Now, by virtue of the Gauss theorem and condition (9) it is

$$\int_F \nabla_1^2 T \, dF = \int_{\Omega_2} \operatorname{grad} T \, . \, d\Omega = \frac{1}{k}\int_{\Omega_2} q \, d\Omega = \frac{q\Omega_2}{k},$$

wherefore

$$c\gamma w \frac{dT_2}{dx} = q \frac{\Omega_2}{F} + k \frac{d^2T_2}{dx^2}. \tag{10}$$

Ω_2 is the perimeter of the surface 2 and approximately (for sufficiently thin sheets) $b\Omega_2 = F$, so that (10) yields

$$c\gamma wb \, . \frac{dT_2}{dx} = q + kb \frac{d^2T_2}{dx^2}. \tag{11}$$

A subsequent simplication consists in assumption of F_1 being a black surface ($\epsilon_1 = 1$, $R_1 = 0$) and in neglection of heat conduction in the body 2 in the x-direction. In such cases the equations, describing the problem, simplify, namely

$$\left.\begin{aligned} q &= \textstyle\int_{F_1} \Delta J \, . \, \phi \, dF_1, \\ \Delta J &= E_1 - E_2 - R_2 \textstyle\int_{F_1} E_1 \phi \, dF_1, \\ c\gamma wb \frac{dT_2}{dx} &= q. \end{aligned}\right\} \tag{12}$$

Besides it is

$$\left.\begin{aligned} E_1 &= \sigma T_1^4 = \text{const.}, \\ E_2 &= \epsilon_2 \sigma T_2^4, \\ (T_2)_{x=0} &= T_{20}. \end{aligned}\right\} \tag{13}$$

Therefore

$$\textstyle\int_{F_1} E_1 \phi \, dF_1 = E_1 \int_{F_1} \phi \, dF_1 = \phi_{1-2} \, . \, E_1,$$

where

$$\phi_{1-2} = \int_{F_1} \frac{\cos\beta_1 \, . \, \cos\beta_2}{\pi r^2} \, dF_1 \tag{14}$$

is the local configuration factor.

Elimination of q and ΔJ from (12) yields

$$c\gamma wb \frac{dT_2}{dx} = E_1 (1 - R_2\phi_{1-2}) \phi_{1-2} - \int_{F_1} \phi E_2 \, dF_1. \tag{15}$$

This equation is an ordinary differential one, since dF_1 does not depend upon x, which—on the other hand—influences E_2; therefore

$$\textstyle\int_{F_1} \phi E_2 \, dF_1 = E_2 \, . \, \phi_{1-2},$$

and

$$c\gamma wb \frac{dT_2}{dx} = E_1 (1 - R_2\phi_{1-2}) \phi_{1-2} - E_2 \phi_{1-2}, \tag{16}$$

or

$$c\gamma wb \frac{dT_2}{dx} = \sigma T_1^4 (1 - R_2\phi_{1-2}) \phi_{1-2} - \phi_{1-2}\sigma\epsilon_2 T_2^4,$$

$$T_2(0) = T_{20}. \tag{17}$$

For example a plane problem will be solved. It is (see Fig. 1) $\beta_1 = \beta_2 = \beta$ and consecutively

$$\phi_{1-2} = \frac{1}{2}\left[1 + \frac{x}{\sqrt{(l^2 + x^2)}}\right]. \tag{18}$$

Assuming $A_2 = \epsilon_2$ and $R_2 = 1 - A_2 = 1 - \epsilon_2$ we obtain

$$m\frac{d\vartheta}{dx^+} = -\epsilon_2\left[1 + \frac{x^+}{\sqrt{(1 + x^{+2})}}\right]\vartheta^4 + \frac{1}{2}\left\{\frac{1}{1 + x^{+2}} + \epsilon_2\left[1 + \frac{x^+}{\sqrt{(1 + x^{+2})}}\right]^2\right\},$$

$$\vartheta(0) = \vartheta_0, \tag{19}$$

where

$$\vartheta = \frac{T_2}{T_1}, \quad \vartheta_0 = \frac{T_{20}}{T_1}, \quad x^+ = \frac{x}{l}, \quad m = \frac{2c\gamma wb}{l\sigma T_1^3}. \tag{20}$$

As a rule, numerical methods must be used in order to obtain the solution of (19). For sufficiently great x^+, however, it may be assumed

$$\frac{1}{1 + x^{+2}} \approx 0, \qquad \frac{x^+}{\sqrt{(1 + x^{+2})}} \approx 1.$$

This approximation is valid if $x^+ > 10$. We get then

$$m\frac{d\vartheta}{dx^+} = 2\epsilon_2(1 - \vartheta^4)$$

and

$$x^+ = \frac{m}{4\epsilon_2}\left(\frac{1}{2}\ln\frac{1 + \vartheta}{1 - \vartheta} + \operatorname{arc\,tg}\vartheta + \text{const.}\right), \tag{21}$$

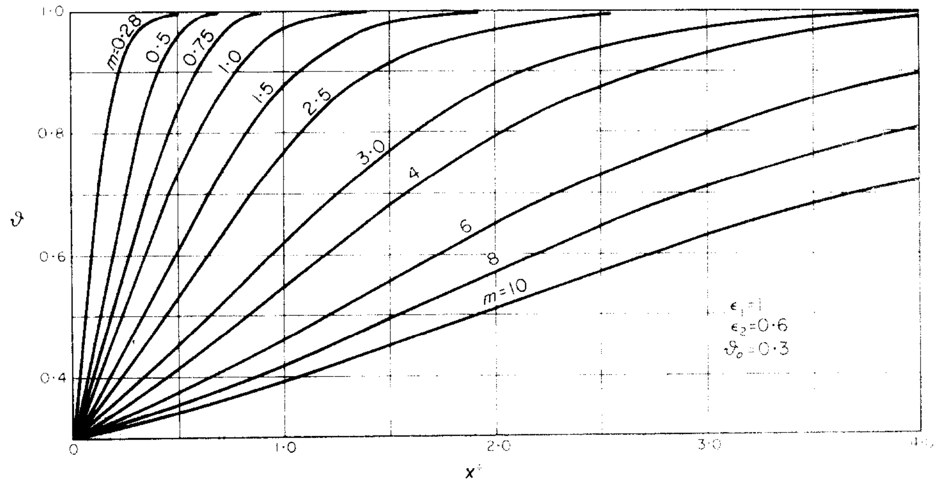

FIG. 2.

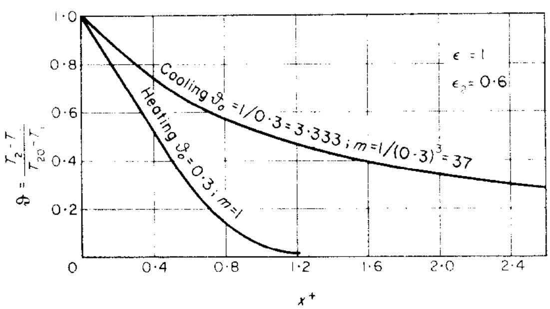

FIG. 3.

so that $x^+ \to \infty$ for $\vartheta \to 1$.

Furthermore, if $x^+ \ll 1$, so that

$$\frac{1}{1 + x^{+2}} \approx 1, \qquad \frac{x^+}{\sqrt{(1 + x^{+2})}} \approx 0,$$

it may be assumed

$$m\frac{\mathrm{d}\vartheta}{\mathrm{d}x^+} = \frac{\epsilon_2 + 1}{2} - \epsilon_2\vartheta^4,$$

and consequently

$$x^+ = \frac{m}{2\epsilon_2 n^3}\left[\frac{1}{2}\ln\left(\frac{n+\vartheta}{n-\vartheta}\cdot\frac{n-\vartheta_0}{n+\vartheta_0}\right) + \text{arc tg}\,\frac{\vartheta-\vartheta_0}{n+(\vartheta\vartheta_0/n)}\right], \quad n^4 = \frac{1+\epsilon_2}{2\epsilon_2}, \tag{22}$$

for small x^+ including $x^+ = 0$.

In Fig. 2 a plot of curves $\vartheta(x^+)$ is given as an example for some values of m; it was assumed $\vartheta_0 = 0{\cdot}3$ and $\epsilon_2 = 0{\cdot}6$. Fig. 3 shows another relationship, namely

$$\theta = \frac{T_2 - T_1}{T_{20} - T_1} = \theta(x^+) \tag{23}$$

for $\epsilon_1 = 1{\cdot}0$; $\epsilon_2 = 0{\cdot}6$. The curves are calculated for conditions of heating and cooling with the same initial temperature difference and the same lowest temperature, *ceteris paribus*, that is for the same value of $(2c\gamma wb)/(l\sigma)$. For the conditions of heating of the surface F_2 it was assumed $\vartheta_0 = 0{\cdot}3$ and $m = 1$; for cooling it was consequently $\vartheta_0 = 1/0{\cdot}3 = 3{\cdot}333$ and $m = 1/(0{\cdot}3)^3 \approx 37$. It can be seen that the temperature distribution in surface F_2 is different in both cases.

To calculate the mean heat flux q_m we first find the heat supplied to the body 2 on the distance x; it is evidently $\dot{Q} = c\gamma wb(T_2 - T_{20})$, whence

$$q_m = \frac{\dot{Q}}{x} = \frac{c\gamma wb}{x}(T_2 - T_{20}),$$

or with introduction of (20)

$$\frac{q_m}{\epsilon_2 E_1} = \frac{m}{2\epsilon_2}\cdot\frac{\vartheta - \vartheta_0}{x^+}. \tag{24}$$

Fig. 4 shows the relationship $q_m/(\epsilon_2 E_1)$ vs.

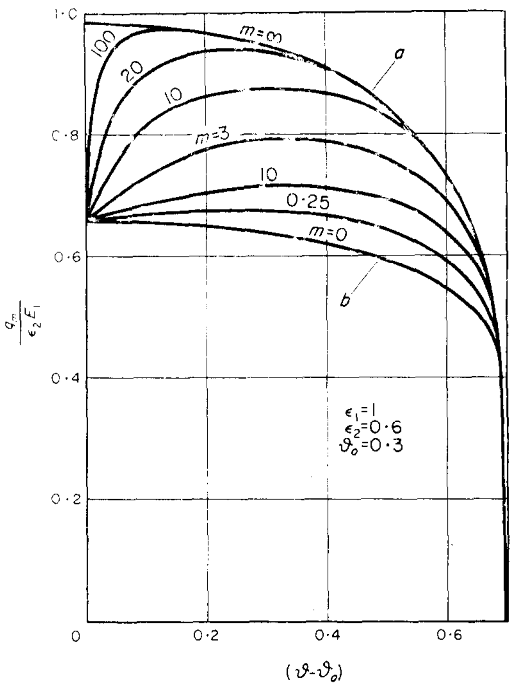

FIG. 4.

$(\vartheta - \vartheta_0)$ for $\epsilon_2 = 0{\cdot}6$; $\vartheta_0 = 0{\cdot}3$. For $x^+ = 0$ we find using (22)

$$\left(\frac{q_m}{\epsilon_2 E_1}\right)_{x^+=0} = \frac{1}{2}\left(\frac{1+\epsilon_2}{2\epsilon_2} - \vartheta_0^4\right). \tag{25}$$

The derivative

$$\frac{d}{d\vartheta}\left(\frac{q_m}{\epsilon_2 E_1}\right)$$

for $x^+ = 0$ may be evaluated by use of (19). Namely it is

$$\vartheta = \vartheta_0 + \vartheta_0' \cdot x^+ + \vartheta_0'' \frac{x^{+2}}{2!} + \vartheta_0''' \cdot \frac{x^{+3}}{3!} + \ldots$$

wherefore

$$\frac{\vartheta - \vartheta_0}{x^+} = \vartheta_0' + \vartheta_0'' \cdot \frac{x^+}{2!} + \vartheta_0''' \cdot \frac{x^{+2}}{3!} + \ldots$$

and

$$\frac{d}{d\vartheta}\left(\frac{q_m}{\epsilon_2 E_1}\right) = \frac{m}{2\epsilon_2}\left(\frac{\vartheta_0''}{2} + \vartheta_0''' \cdot \frac{x^+}{3} + \ldots\right) \cdot \frac{dx^+}{d\vartheta}$$

$$= \frac{m}{2\epsilon_2} \cdot \frac{\dfrac{\vartheta_0''}{2} + \dfrac{\vartheta_0'''}{3} x^+ + \ldots}{\vartheta_0' + \vartheta_0'' \cdot x^+ + \dfrac{\vartheta_0'''}{2} \cdot x^{+2} + \ldots}$$

Therefore

$$\left[\frac{d}{d\vartheta}\left(\frac{q_m}{\epsilon_2 E_1}\right)\right]_{x^+=0} = \frac{m}{4\epsilon_2} \cdot \frac{\vartheta_0''}{\vartheta_0'}.$$

The values of ϑ_0' and ϑ_0'' may be evaluated from (19). The final result is

$$\left[\frac{d}{d\vartheta}\left(\frac{q_m}{\epsilon_2 E_1}\right)\right]_{x^+=0} = -\vartheta_0^3 + \frac{m}{4\epsilon_2} \cdot \frac{1 - \vartheta_0^4}{\dfrac{1+\epsilon_2}{2\epsilon_2} - \vartheta_0^4}. \tag{26}$$

Thus it can be seen that the curve $m =$ const. in Fig. 4 approaches the sooner the curve *a*, valid for $x^+ \gg 1$, the greater is m; the curves with $m \ll 1$ differ very little from the curve *b*, valid for $x^+ \ll 1$. Therefore the curve *a*, obtained from (21), may be regarded as a curve $m =$ const. for $m = \infty$; the curve *b*, which is obtained from (22), is valid for $m = 0$.

Résumé—Cet article donne les équations du rayonnement thermique entre des surfaces en mouvement relatif. A tite d'exemple le cas des plaques parallèles est étudié. La solution est obtenue par des méthodes numériques.

Zusammenfassung—Die Gleichungen für den Wärmeübergang durch Strahlung zwischen Oberflächen, die sich relativ zueinander bewegen, werden hier abgeleitet. Ein spezialler Fall mit parallelen Platten ist an einem Beispiel erläutert. Die Lösung lässt sich mit numerischen Methoden erhalten.